Upscaling for the time-harmonic Maxwell equations with heterogeneous magnetic materials

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This paper presents a theoretical method for the upscaling of the time-harmonic Maxwell equations. We use the eddy current approximation of the Maxwell equations to describe the fields in heterogeneous materials. The magnetic permeability of the media is assumed to have random heterogeneities given by a Gaussian random field. The upscaling is based on the coarse graining method which applies projections and Green's function formalism in Fourier space to scale the electric field. An upscaled Maxwell equation is derived which includes an effective magnetic permeability tensor. The effective permeability explicitly depends on the given scale for the upscaling. The scale-dependent permeability is calculated by a second-order perturbative expansion, and we discuss the future verification and the application of the results.

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I. INTRODUCTION

Eddy currents in heterogeneous magnetic materials are induced by the electric field in the given media. The electric field and, therefore, the eddy currents strongly depend on the heterogeneity of the magnetic permeability of the materials. The quantification of the eddy currents and the resulting electric energy loss due to dissipation is very interesting for engineering applications. To make a reliable prediction of the energy loss on the macroscopic scale, however, it is essential to incorporate the fine-scale structure of the materials. Stochastic models are a valuable tool to analyze the electric field in such materials. In the stochastic approach the heterogeneities of the media are modeled as a time-independent random field with given statistical properties. The characteristic macroscopic behavior then follows from appropriately defined averages over the ensemble of all possible material realizations. Such an approach has been used in the past to determine effective dispersion coefficients for the macroscopic scale behavior in transport theory, see, e.g., [1,2] for an overview. Since experimental results of macroscopic permeabilities depend on the resolution scale at which the measurements are done, it is important to study the impact of the resolution scale on the electric field. However, the study of scale-dependent magnetic permeability with the help of the stochastic approach has been rarely focused on in the past.

For a long time the method of homogenization has been used to get averaged or so-called homogenized equations for various differential operators. This idea has been brought onto the Maxwell equations by Jikov *et al.* and Bossavit [3,4]. Therein the homogenized electric and magnetic properties of a material with a periodic microstructure have been found from the solution of a local problem on the unit cell by suitable averages. However, these studies are restricted to the scaling of the Maxwell equations in periodic structures from the view of a two-scale analysis, see also [5]. Recent studies on homogenization of the Maxwell equations have also focused on properties of heterogeneous composite materials, see [6–9]. A scaling theory for homogenization of the Maxwell equations has been developed for such materials by Vinogradov and Aivazyan [7]. In [8,9] the homogenization of the Maxwell equations at fixed frequency has been addressed, and the work of Dular *et al.* [6] investigated the homogenization problem for thin layers by a magnetic vector potential formulation. They combined for the first time the eddy current approximation and homogenization in lamination stacks. Further developments of upscaling by the homogenization for laminated steel and anisotropic or amorphous cores has been done by Bergqvist and Engdahl [10], Kiwitt *et al.* [11], and Shin and Lee [12]. However, due to the assumption of periodicity, these studies are of limited prediction for stochastic heterogeneous media. Moreover, the homogenized results represent the physical behavior only for a global upscaling without any possibility for an upscaling to arbitrary intermediate length scales. Although the homogenization of the Maxwell equations has been well analyzed in the literature, there are no studies which directly focus on the scale-dependent upscaling of the Maxwell equations or the eddy current approximation.

In this paper we use the theoretical concept provided by the coarse graining method to upscale the electric field in the eddy current approximation of the Maxwell equations. In cases of upscaling of flow and transport processes in heterogenous media the coarse graining method proved very useful, see [13–15]. As shown in [13], this method predicts the exact scale-dependent transition for the effective hydraulic permeability. We derive for the first time an upscaled equation for the electric field by the coarse graining method. The method yields a scale-dependent effective permeability by upscaling the heterogeneous magnetic material. The upscaled magnetic permeability handles the scale transition from the microscale to macroscale. By a perturbative expansion we obtain explicit results for the upscaled permeability. In the Appendixes we extend the coarse graining method toward a numerical upscaling to be able to compute local upscaled permeabilities. So far, the coarse graining method has been merely used for the upscaling of scalar equations. The extension of this method for the upscaling of vector fields is also

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new. The results of the upscaling can be applied to predict permeabilities of effective media in materials physics, such as composite metamaterials or granular ferromagnetic metals, as well as to compute upscaled magnetic permeabilities for aggregated grid elements in numerical simulations.

The conclusion of this brief review is that the effective magnetic permeability is in general a tensor which reduces to a scalar in isotropic media. The effective coefficients show an increase with the resolution scale and adopt asymptotic values for scales larger than ten correlation lengths. In the anisotropic case, the asymptotic values depend strongly on the correlation lengths.

II. EDDY CURRENT APPROXIMATION OF THE MAXWELL EQUATIONS

For the derivation of the upscaling we consider the three-dimensional (3D) Euclidean space $\mathbb{R}^3$ in which the electromagnetic problem is defined. In the time-harmonic approach the electric field is given by $\mathcal{E}(\boldsymbol{x},t)=\mathrm{Re}[\boldsymbol{E}(\boldsymbol{x})\exp(iwt)]$, the magnetic field by $\mathcal{H}(\boldsymbol{x},t)=\mathrm{Re}[\boldsymbol{H}(\boldsymbol{x})\exp(iwt)]$, and the current is given by $\mathcal{J}(\boldsymbol{x},t)=\mathrm{Re}[\boldsymbol{j}(\boldsymbol{x})\exp(iwt)]$, $w\geqslant 0$. Due to the eddy current approximation the system is then described by the equations (see, e.g., Bossavit [16])

$$\mathrm{curl}\,\boldsymbol{H}(\boldsymbol{x})=\boldsymbol{j}(\boldsymbol{x}),$$

$$\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x})=-i\omega\mu(\boldsymbol{x})\boldsymbol{H}(\boldsymbol{x})$$

for $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$. The magnetic permeability is denoted by μ and the electric conductivity by σ in the following. Further, we assume Ohm's law where the current density is given by $\boldsymbol{j}=\sigma\boldsymbol{E}+\boldsymbol{j}^G$ with a generator current $\boldsymbol{j}^G$. In the electric formulation we then get the second-order partial differential equation

$$\mathrm{curl}\,\mu^{-1}(\boldsymbol{x})\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x})+i\omega\sigma\boldsymbol{E}(\boldsymbol{x})=-i\omega\boldsymbol{j}^G(\boldsymbol{x}). \tag{1}$$

In the following we use the identities $\mathrm{curl}\,\mathrm{curl}=\mathrm{grad}\,\mathrm{div}-\Delta$ and $[\mathrm{curl}\,a]_i=\epsilon_{ijk}\partial_j a_k$, where ϵ_{ijk} denotes the asymmetric Levi-Civita tensor and $\partial_j=\partial/\partial x_j$. The heterogeneous magnetic permeability $\mu(\boldsymbol{x})$ is taken as a scalar field given by a Gaussian random field. We define the inverse magnetic permeability ν by $\nu(\boldsymbol{x}):=1/\mu(\boldsymbol{x})$ where we assume $0<\mu(\boldsymbol{x})<\infty$.

A. Correlation function

In the stochastic approach the spatially inhomogeneous distribution $\nu(\boldsymbol{x})$ is identified with a single realization of a stochastic process defined by the ensemble of all possible realizations. We assume this process to be statistically translation invariant in space which implies that the ensemble average $\overline{\nu(\boldsymbol{x})}$ does not depend on the position $\boldsymbol{x}$. The overbar always stands for the average over the ensemble. We split the field into its deterministic mean and a random contribution,

$$\nu(\boldsymbol{x})=\bar{\nu}+\tilde{\nu}(\boldsymbol{x}), \tag{2}$$

where $\bar{\nu}=\overline{\nu(\boldsymbol{x})}$ is a constant value and $\tilde{\nu}(\boldsymbol{x})$ is the fluctuation field. For a Gaussian random field, $\tilde{\nu}(\boldsymbol{x})$ is a random function with zero mean. We define $\overline{\hat{\tilde{\nu}}(\boldsymbol{k})\hat{\tilde{\nu}}(\boldsymbol{k}')}=(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')\bar{\nu}^2\hat{C}(\boldsymbol{k})$. The Fourier transform is defined by $\hat{f}(\boldsymbol{k})=\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}f(\boldsymbol{x})$ and $f(\boldsymbol{x})=\int_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{f}(\boldsymbol{k})$, where $\int_k\cdots\equiv(2\pi)^{-3}\int d^3k\cdots$ is a shorthand notation.

The function $\hat{C}(\boldsymbol{k})$ denotes the autocorrelation spectrum of the inverse magnetic permeability. The approach is mathematically well-defined subject to some additional requirements for C. The particular functional form of C is to some extent arbitrary. Reflecting the situation in a heterogeneous material, it should drop to zero sharply for lengths larger than the intrinsic correlation length scales l_i. We choose a Gauss-shaped function for C. Thus the autocorrelation spectrum $\hat{C}(\boldsymbol{k})$ is given by $\hat{C}(\boldsymbol{k})=q_0(2\pi)^{3/2}\prod_{i=1}^3 l_i\exp(-k_i^2l_i^2/2)$.

The variance q_0 measures the strength of the heterogeneities, and l_i, $i=1,\ldots,3$, denotes the correlation length of the field in the direction of x_i. In the anisotropic case two or more correlation lengths are unequal. For an isotropic field the lengths l_i are equally denoted by l_0. In that case, the correlation function merely depends on the distance $|\boldsymbol{x}-\boldsymbol{x}'|$.

B. Green's function in Fourier space

Due to Eqs. (1) and (2) the resulting equation for the electric field reads

$$\bar{\nu}\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x})+i\omega\sigma\boldsymbol{E}(\boldsymbol{x})=-i\omega\boldsymbol{j}^G(\boldsymbol{x})-\mathrm{curl}\,\tilde{\nu}(\boldsymbol{x})\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}) \tag{3}$$

which yields using $\mathrm{div}\,\boldsymbol{E}=\rho$

$$-\bar{\nu}\Delta\boldsymbol{E}(\boldsymbol{x})+i\omega\sigma\boldsymbol{E}(\boldsymbol{x})=-i\omega\boldsymbol{j}^G(\boldsymbol{x})-\bar{\nu}\,\mathrm{grad}\,\rho(\boldsymbol{x})-\mathrm{curl}\,\tilde{\nu}(\boldsymbol{x})\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}). \tag{4}$$

We define $\gamma:=-i\omega\boldsymbol{j}^G-\bar{\nu}\,\mathrm{grad}\,\rho$, and we use Einstein's sum convention for the rest of the paper. Considering the ith component of Eq. (4),

$$-\bar{\nu}\sum_{j=1}^{3}\partial_j^2E_i(\boldsymbol{x})+i\omega\sigma E_i(\boldsymbol{x})=\gamma_i(\boldsymbol{x})-\epsilon_{ijk}\epsilon_{klm}\partial_j\tilde{\nu}(\boldsymbol{x})\partial_lE_m(\boldsymbol{x}),$$

the Fourier transform yields

$$-\bar{\nu}\sum_{j=1}^{3}(ik_j)^2\hat{E}_i(\boldsymbol{k})+i\omega\sigma\hat{E}_i(\boldsymbol{k})=\hat{\gamma}_i(\boldsymbol{k})-\epsilon_{ijk}\epsilon_{klm}ik_j\int_{k'}\hat{\tilde{\nu}}(\boldsymbol{k}-\boldsymbol{k}')ik_l'\hat{E}_m(\boldsymbol{k}'). \tag{5}$$

Equation (5) can be rewritten as

$$\hat{E}_i(\boldsymbol{k})=g_0(\boldsymbol{k})\hat{\gamma}_i(\boldsymbol{k})-\epsilon_{ijk}\epsilon_{klm}g_0(\boldsymbol{k})\int_{k'}R_{jl}(\boldsymbol{k},\boldsymbol{k}')\hat{E}_m(\boldsymbol{k}') \tag{6}$$

with the definitions $R_{jl}(\boldsymbol{k},\boldsymbol{k}'):=ik_j\hat{\tilde{\nu}}(\boldsymbol{k}-\boldsymbol{k}')ik_l'$ and $g_0(\boldsymbol{k}):=(\bar{\nu}\boldsymbol{k}^2+i\omega\sigma)^{-1}$. We define Green's function $\hat{G}(\boldsymbol{k},\boldsymbol{k}')$ of Eq. (6) by

$$\int_{k''}[g_0(\boldsymbol{k}'')^{-1}\delta_{im}\delta(\boldsymbol{k}-\boldsymbol{k}'')+\epsilon_{ijk}\epsilon_{klm}R_{jl}(\boldsymbol{k},\boldsymbol{k}'')]\hat{G}_{im}(\boldsymbol{k}'',\boldsymbol{k}') = \delta(\boldsymbol{k}+\boldsymbol{k}') \tag{7}$$

with summation over the indices j,k,l,m, where the index i is fixed. For $R_{jl}\equiv 0$, we obtain for the ith component of the tensorial Green's function $\hat{G}_{ii}(\boldsymbol{k},\boldsymbol{k}')=g_0(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')$. Hence $\hat{G}$ is diagonal for $\hat{\tilde{\nu}}\equiv 0$.

C. Projections in Fourier space

We define projections for cutting off high and low frequency modes of the Fourier transformed $\hat{\boldsymbol{E}}$ by

$$P^-_{\lambda,k}(\hat{\boldsymbol{E}}(\boldsymbol{k})) := \begin{cases}\hat{\boldsymbol{E}}(\boldsymbol{k}) & \text{if } |k_i| \leqslant a_s/\lambda \text{ for all } i\in\{1,2,3\}\\ 0 & \text{otherwise,}\end{cases}$$

$$P^+_{\lambda,k}(\hat{\boldsymbol{E}}(\boldsymbol{k})) := \begin{cases}\hat{\boldsymbol{E}}(\boldsymbol{k}) & \text{if } |k_i| > a_s/\lambda \text{ for an } i\in\{1,2,3\}\\ 0 & \text{otherwise,}\end{cases}$$

where $P^+_{\lambda,k}[P^+_{\lambda,k}(\hat{\boldsymbol{E}}(\boldsymbol{k}))]=P^+_{\lambda,k}(\hat{\boldsymbol{E}}(\boldsymbol{k}))$, and for $P^-_{\lambda,k}$ analogously. The parameter $a_s\geqslant 1$ is a constant. If it is possible we will omit the index λ and k in the following and use $P^+_{\lambda,k,k'}(\hat{\boldsymbol{E}}(\boldsymbol{k},\boldsymbol{k}'))$ instead of $P^+_{\lambda,k}[P^+_{\lambda,k'}(\hat{\boldsymbol{E}}(\boldsymbol{k},\boldsymbol{k}'))]$. Further, we define the following operators

$$\mathcal{L}_i(\hat{\boldsymbol{E}}) := \int L_{im}(\boldsymbol{k},\boldsymbol{k}')\hat{E}_m(\boldsymbol{k}')d^3k' := \int [g_0(\boldsymbol{k}')^{-1}\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{im} + \epsilon_{ijk}\epsilon_{klm}R_{jl}(\boldsymbol{k},\boldsymbol{k}')]\hat{E}_m(\boldsymbol{k}')d^3k'$$

$$\mathcal{R}_i(\hat{\boldsymbol{E}}) := -\epsilon_{ijk}\epsilon_{klm}\int R_{jl}(\boldsymbol{k},\boldsymbol{k}')\hat{E}_m(\boldsymbol{k}')d^3k'.$$

With the definition of L_{im} we obtain $\int_{k''}L_{im}(\boldsymbol{k},\boldsymbol{k}'')\hat{G}_{im}(\boldsymbol{k}'',\boldsymbol{k}')=\delta(\boldsymbol{k}+\boldsymbol{k}')\delta_{ii}$ (i fixed) for Green's function, that is, $L^{-1}_{im}(\boldsymbol{k},\boldsymbol{k}'):=(2\pi)^{-6}\hat{G}_{im}(\boldsymbol{k},-\boldsymbol{k}')$. Further we apply Eq. (7) for the projected Green's function which entails

$$\int_{k''}P^+_{\lambda,k}(L_{im}(\boldsymbol{k},\boldsymbol{k}''))P^+_{\lambda,k'}(\hat{G}_{im}(\boldsymbol{k}'',\boldsymbol{k}'))=P^+_{\lambda,k,k'}(\delta(\boldsymbol{k}+\boldsymbol{k}'))\delta_{ii},$$

and, therefore $[P^+_{\lambda,k'}(L_{im}(\boldsymbol{k}'',\boldsymbol{k}'))]^{-1}:=(2\pi)^{-6}P^+_{\lambda,k'}(\hat{G}_{im}(\boldsymbol{k}'',-\boldsymbol{k}'))$. The latter leads to the definition

$$(P^+\mathcal{L})^{-1}_i(\hat{\boldsymbol{E}}) := \frac{1}{(2\pi)^3}\int_{k'}P^+_{\lambda,k'}\hat{G}_{im}(\boldsymbol{k},-\boldsymbol{k}')\hat{E}_m(\boldsymbol{k}').$$

We remark that Green's function fulfills $\mathcal{L}G_i=\delta e_i$ where G_i stands for the column vector $(G_{im})_{m=1,2,3}$. The summation over the index m is given in $\mathcal{L}G_i$.

III. COARSE GRAINING METHOD

We develop the upscaling for the eddy current approximation of the time-harmonic Maxwell equation using the coarse graining method. This method was developed and applied in [13,14,17] to scale the flow and transport equation for heterogeneous media. The upscaling is based on filtering in Fourier space, i.e., high oscillatory modes are eliminated by cutting off the function values of the Fourier transform for large wave vectors. The upscaling results in an upscaled equation on larger scales starting from the process for the fine-scale media. The coarse graining method does not model the fine-scale heterogeneity up to the given length scale explicitly, but models the influences of the subscale fluctuations by a scale-dependent parameter which incorporates the impact of the unresolved fluctuations. In the following we denote the coarser scale by λ and omit the Fourier variables $\boldsymbol{k}$ and $\boldsymbol{k}'$.

According to Eq. (6) and the definition of $\mathcal{R}$ the electric field is given by

$$\hat{E}_i = g_0\hat{\gamma}_i + g_0\mathcal{R}_i(\hat{\boldsymbol{E}}) \tag{8}$$

which leads to the equation

$$\mathcal{L}_i(\hat{\boldsymbol{E}}) = \hat{\gamma}_i. \tag{9}$$

Further, it is obvious that

$$\hat{E}_i = P^+_{\lambda,k}(\hat{E}_i) + P^-_{\lambda,k}(\hat{E}_i) \tag{10}$$

holds true due to the projections, see Sec. II C. From Eqs. (8) and (9) we get $\mathcal{L}_i(\hat{\boldsymbol{E}})=g_0^{-1}\hat{E}_i-\mathcal{R}_i(\hat{\boldsymbol{E}})$ which leads to

$$P^+[\mathcal{L}_i(P^+(\hat{\boldsymbol{E}}))] = P^+[g_0^{-1}P^+(\hat{E}_i)-\mathcal{R}_i(P^+(\hat{\boldsymbol{E}}))] = P^+(\hat{\gamma}_i)+P^+(\mathcal{R}_iP^-(\hat{\boldsymbol{E}})).$$

Using the operator $(P^+\mathcal{L})^{-1}_i$ defined in Sec. II C, we get for $\hat{E}_i$

$$P^+(\hat{E}_i) = (P^+\mathcal{L})^{-1}_i[P^+(\hat{\gamma})+P^+(\mathcal{R}P^-(\hat{\boldsymbol{E}}))] = (P^+\mathcal{L})^{-1}[P^+(\hat{\gamma})]+(P^+\mathcal{L})^{-1}[P^+(\mathcal{R}P^-(\hat{\boldsymbol{E}}))]. \tag{11}$$

We project Eq. (8) with the aid of P^- and insert then Eq. (10) in the right-hand side, where $P^+_{\lambda,k}(\hat{E}_i)$ is replaced by Eq. (11), which yields

$$P^-(\hat{E}_i) = P^-[g_0\hat{\gamma}_i + g_0\mathcal{R}_i(P^-(\hat{\boldsymbol{E}}))] + P^-[g_0\mathcal{R}_iP^+((P^+\mathcal{L})^{-1}(P^+(\hat{\gamma})))] + P^-[g_0\mathcal{R}_i[P^+((P^+\mathcal{L})^{-1}(P^+(\mathcal{R}P^-(\hat{\boldsymbol{E}}))))]]. \tag{12}$$

Under the assumption that $\hat{\gamma}$ is given on the macroscopic scale, i.e., $P^+\hat{\gamma}\equiv 0$, the second term of Eq. (12) vanishes. As a result the equation for $P^-(\hat{E}_i)$ can be rewritten as

$$P^-(g_0^{-1}\hat{E}_i) = S_i(\boldsymbol{k}) + Q_i(\boldsymbol{k}) \tag{13}$$

where

$$S_i(\boldsymbol{k}) := P_k^-\left(\hat{\gamma}_i(\boldsymbol{k}) - \epsilon_{ijk}\epsilon_{klm}\int R_{jl}(\boldsymbol{k},\boldsymbol{k}')P^-(\hat{E}_m(\boldsymbol{k}'))d^3k'\right).$$

The expression for Q is given by

$$Q_i(\boldsymbol{k}) := P^-[\mathcal{R}_i[P^+((P^+\mathcal{L})^{-1}(P^+(\mathcal{R}P^-(\hat{\boldsymbol{E}}))))]]$$
$$= \frac{1}{(2\pi)^6}P_k^-\left[-\epsilon_{ijk}\epsilon_{klm}\int R_{jl}(\boldsymbol{k},\boldsymbol{k}')P_{k'}^+\left(\int P_{k''}^+\hat{G}_{mn}(\boldsymbol{k}',-\boldsymbol{k}'')\left[-\epsilon_{nqp}\epsilon_{prs}\int R_{qr}(\boldsymbol{k}'',\boldsymbol{k}''')P_{k'''}^-(\hat{E}_s(\boldsymbol{k}'''))d^3k'''\right]d^3k''\right)d^3k'\right].$$

It represents the impact of the heterogeneity of the media of scales smaller than λ. In the following, Q is approximated by its mean $\bar{Q}$ which corresponds to a mean-field approximation as done in [14]. The ensemble mean of Q is given by

$$\overline{Q_i(\boldsymbol{k})} = P_k^-\left[\epsilon_{ijk}\epsilon_{klm}ik_j\int_{k'}\overline{\hat{\tilde{\nu}}(\boldsymbol{k}-\boldsymbol{k}')ik'_lP_{k'}^+\left(\int_{k''}P_{k''}^+\hat{G}_{mn}(\boldsymbol{k}',-\boldsymbol{k}'')\epsilon_{nqp}\epsilon_{prs}ik''_q\hat{\tilde{\nu}}(\boldsymbol{k}''-\boldsymbol{k})ik_rP_k^-(\hat{E}_s(\boldsymbol{k}))\right)}\right]$$
$$= P_k^-(\epsilon_{ijk}ik_j\delta\nu_{kp}^{eff}(\boldsymbol{k},\lambda)ik_r\epsilon_{prs}\hat{E}_s(\boldsymbol{k})), \tag{14}$$

where the last expression defines an inverse effective magnetic permeability tensor given by

$$\delta\nu_{kp}^{eff}(\boldsymbol{k},\lambda) := \epsilon_{klm}\int_{k'}\int_{k''}\overline{\hat{\tilde{\nu}}(\boldsymbol{k}-\boldsymbol{k}')ik'_lP_{k',k'',\lambda}^+(\hat{G}_{mn}(\boldsymbol{k}',-\boldsymbol{k}'')\epsilon_{nqp}ik''_q\hat{\tilde{\nu}}(\boldsymbol{k}''-\boldsymbol{k}))}.$$

The result represents a nonlocal effective material parameter. We apply a localization $\boldsymbol{k}=0$ as done in [13,17], and we obtain

$$\delta\nu_{kp}^{eff}(\lambda) = \epsilon_{klm}\int_{k'}\int_{k''}\overline{\hat{\tilde{\nu}}(-\boldsymbol{k}')ik'_lP_{k',k'',\lambda}^+(\hat{G}_{mn}(\boldsymbol{k}',-\boldsymbol{k}'')\epsilon_{nqp}ik''_q\hat{\tilde{\nu}}(\boldsymbol{k}''))}. \tag{15}$$

In the isotropic case, $\delta\nu_{kp}^{eff}(\lambda)$ reduces to a diagonal tensor.

A. Upscaled time-harmonic model equation

According to Eq. (13) we obtain on the scale λ using Eq. (14)

$$P_\lambda^-(g_0^{-1}E_i) = S_i(\boldsymbol{k}) + \overline{Q_i(\boldsymbol{k})}$$
$$= P_k^-\left(\hat{\gamma}_i(\boldsymbol{k}) - \epsilon_{ijk}\epsilon_{klm}\int R_{jl}(\boldsymbol{k},\boldsymbol{k}') \times P^-(\hat{E}_m(\boldsymbol{k}'))d^3k'\right) + P_k^-(\epsilon_{ijk}ik_j\delta\nu_{kp}^{eff}(\lambda) \times ik_r\epsilon_{prs}\hat{E}_s(\boldsymbol{k})),$$

so that $P_\lambda^-(E_i)$ can be rewritten as

$$P_\lambda^-(E_i) = g_0(\boldsymbol{k})P_k^-(-i\omega\hat{j}_i^G(\boldsymbol{k}) - \bar{\nu}(\widehat{\text{grad}\,\rho})_i) - g_0(\boldsymbol{k})P_k^-\left(\epsilon_{ijk}\epsilon_{klm}ik_j\int\hat{\tilde{\nu}}(\boldsymbol{k}-\boldsymbol{k}')ik'_lP^-(\hat{E}_m(\boldsymbol{k}'))d^3k'\right) + g_0(\boldsymbol{k})P_k^-(\epsilon_{ijk}ik_j\delta\nu^{eff}(\lambda)ik_r\epsilon_{krs}\hat{E}_s(\boldsymbol{k}))$$

where the term $(\widehat{\text{grad}\,\rho})_i$ is induced by $ik_i\hat{\rho}$ in Fourier space. The upscaled equation for the upscaled electric field $\boldsymbol{E}_\lambda(\boldsymbol{x})$ reads in physical space (using the equality $\bar{\nu}\,\text{grad}\,\rho_\lambda - \bar{\nu}\Delta\boldsymbol{E}_\lambda = \text{curl}\,\bar{\nu}\,\text{curl}\,\boldsymbol{E}_\lambda$)

$$\text{curl}\,\bar{\nu}\,\text{curl}\,\boldsymbol{E}_\lambda(\boldsymbol{x}) + i\omega\sigma\boldsymbol{E}_\lambda(\boldsymbol{x}) = -i\omega\boldsymbol{j}_\lambda^G(\boldsymbol{x}) - \text{curl}\,\tilde{\nu}_\lambda(\boldsymbol{x})\text{curl}\,\boldsymbol{E}_\lambda(\boldsymbol{x}) + \text{curl}\,\delta\nu^{eff}(\lambda)\text{curl}\,\boldsymbol{E}_\lambda(\boldsymbol{x}).$$

The upscaled fluctuations $\tilde{\nu}_\lambda(\boldsymbol{x})$ and the upscaled current $\boldsymbol{j}_\lambda^G(\boldsymbol{x})$ are given by the Fourier back transform of $P^-(\hat{\tilde{\nu}})$ and $P^-(\boldsymbol{j}^G)$, respectively. Defining the effective permeability $\mu_{ij}^{eff}(\lambda) := [\bar{\nu} - \delta\nu_{ij}^{eff}(\lambda)]^{-1}$, the upscaled equation for $\boldsymbol{E}_\lambda$ on the scale λ is given by

$$\text{curl}\{[\mu^{eff}(\lambda)]^{-1} + \tilde{\nu}_\lambda(\boldsymbol{x})\}\text{curl}\,\boldsymbol{E}_\lambda(\boldsymbol{x}) + i\omega\sigma\boldsymbol{E}_\lambda(\boldsymbol{x}) = -i\omega\boldsymbol{j}_\lambda^G(\boldsymbol{x}).$$

Comparing the upscaled equation with the equation (1), the effective permeability $\mu^{eff}(\lambda)$ includes the scale behavior from the microscale for $\lambda=0$ to the macroscale for $\lambda\to\infty$.

IV. EXPLICIT RESULTS BY A SECOND-ORDER PERTURBATION THEORY

Explicit results for the scale-dependent inverse permeability $\delta\nu^{eff}$ can be derived by a second-order perturbation theory where Green's function reduces to $\hat{G}_{mn}(\boldsymbol{k},-\boldsymbol{k}') = g_0(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{mn}$. Hence we get from Eq. (15)

$$\delta\nu_{kp}^{eff}(\lambda) = \epsilon_{klm}\int_{k'} \overline{\hat{\tilde{\nu}}(-\boldsymbol{k}')ik_l' P_{k',\lambda}^{+}(\bar{\nu}\boldsymbol{k}'^2 + i\omega\sigma)^{-1}\epsilon_{mqp}ik_q'\hat{\tilde{\nu}}(\boldsymbol{k}')}$$

which can be simplified to

$$\begin{aligned}\delta\nu_{kp}^{eff}(\lambda) &= -\int_{k'} (\delta_{kq}\delta_{lp} \\ &\quad - \delta_{kp}\delta_{ql})\overline{\hat{\tilde{\nu}}(-\boldsymbol{k}')k_l'k_q'P_{k',\lambda}^{+}(\bar{\nu}\boldsymbol{k}'^2 + i\omega\sigma)^{-1}\hat{\tilde{\nu}}(\boldsymbol{k}')} \\ &= -\int_{k'} P_{k',\lambda}^{+}\left(\frac{\overline{\hat{\tilde{\nu}}(-\boldsymbol{k}')\hat{\tilde{\nu}}(\boldsymbol{k}')}}{\bar{\nu}\boldsymbol{k}'^2 + i\omega\sigma}\right)(k_k'k_p' - \delta_{kp}\boldsymbol{k}'^2). \quad (16)\end{aligned}$$

Analogous to [13,17] we introduce a smooth cutoff function instead of $P_{k,\lambda}^{+}$ for the calculations of $\delta\nu_{kp}^{eff}(\lambda)$ by

$$P_{k,\lambda}^{+} \to \left[1 - \exp\left(-\frac{\boldsymbol{k}^2\lambda^2}{2a_w^2}\right)\right]. \quad (17)$$

The parameter $a_w \geq 1$ is a constant.

A. Isotropic case

We state the results for the isotropic case of a three-dimensional system with isotropic correlations function. In the given second-order approximation we obtain that the effective inverse magnetic permeability is a scalar, i.e., $\delta\nu_{kp}^{eff} = \delta\nu^{eff}\delta_{kp}$, where

$$\begin{aligned}\delta\nu^{eff}(\lambda) &= \frac{2q_0 l_0^3\bar{\nu}^2}{3(2\pi)^{3/2}}\int \frac{\boldsymbol{k}^2}{\bar{\nu}\boldsymbol{k}^2 + i\omega\sigma}(\exp(-\boldsymbol{k}^2 l_0^2/2) \\ &\quad - \exp\{-\boldsymbol{k}^2[l_0^2/2 + \lambda^2/(2a_w^2)]\})d^3k \\ &= \frac{4q_0 l_0^3\bar{\nu}}{3\sqrt{2\pi}}[M(l_0^2/2;\omega\sigma/\bar{\nu}) - M(l_0^2/2 + \lambda^2/(2a_w^2);\omega\sigma/\bar{\nu})]\end{aligned} \quad (18)$$

using Eq. (17). The function $M(a;b)$ is defined by

$$\begin{aligned}M(a;b) &:= \int_0^\infty \frac{x^4\exp(-ax^2)}{x^2 + ib}dx \\ &= \frac{\sqrt{\pi}}{4a^{3/2}} - \frac{i\sqrt{\pi}b}{2\sqrt{a}} \\ &\quad + \frac{\pi b^{3/2}}{2}(-1)^{3/4}\exp(iab)\mathrm{erfc}(\sqrt{iab}).\end{aligned}$$

The error function $\mathrm{erfc}(x)$ is defined as given by [18]. In Fig. 1 the scale-dependent behavior of $\mu^{eff}(\lambda)$ is shown for different isotropic correlation lengths l_0 for conductivity $\sigma=0$.

B. Anisotropic case

In the case of an anisotropic correlation function the result for the effective inverse permeability no longer reduces to standard integrals. From Eq. (16) we get using Eq. (17)

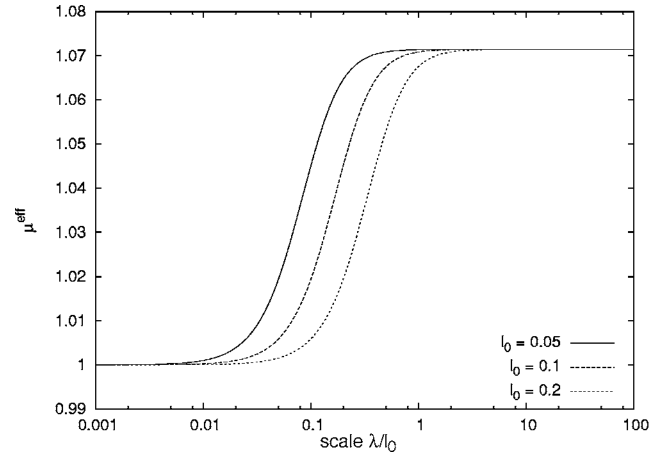

FIG. 1. Scale-dependent behavior of $\mu^{eff}(\lambda)$ in the isotropic case for $\sigma=0$, $\bar{\nu}=1$, $q_0=0.1$, $a_w=2$, and different correlation lengths l_0.

$$\begin{aligned}\delta\nu_{kp}^{eff}(\lambda) &= -q_0\bar{\nu}^2(2\pi)^{3/2}\int\left[1 - \exp\left(-\frac{\boldsymbol{k}^2\lambda^2}{2a_w^2}\right)\right] \\ &\quad \times\frac{\prod_{i=1}^3 l_i\exp(-k_i^2l_i^2/2)}{\bar{\nu}\boldsymbol{k}^2 + i\omega\sigma}(k_kk_p - \delta_{kp}k^2)d^3k \\ &= -q_0\bar{\nu}^2(2\pi)^{3/2}\int\int_0^\infty d\Lambda\exp[-\Lambda(\bar{\nu}\boldsymbol{k}^2 + i\omega\sigma)] \\ &\quad \times\left[1 - \exp\left(-\frac{\boldsymbol{k}^2\lambda^2}{2a_w^2}\right)\right](k_kk_p - \delta_{kp}k^2)\prod_{i=1}^3 l_i \\ &\quad \times\exp(-k_i^2l_i^2/2)d^3k.\end{aligned} \quad (19)$$

To derive an approximate result for $\delta\nu_{kp}^{eff}$ we expand the integrand of Eq. (19) with respect to σ and consider only the leading-order term for $\sigma\to 0$. Thus we expand the exponential function of the integrand by $\exp(-\Lambda i\omega\sigma)\approx 1-\Lambda i\omega\sigma$, and we account for the constant term only. The inverse permeability reads then in lowest-order of σ

$$\begin{aligned}\delta\nu_{kp}^{eff}(\lambda) &= -q_0\bar{\nu}l_1l_2l_3(2\pi)^{3/2}\int_k\left[1 - \exp\left(-\frac{\boldsymbol{k}^2\lambda^2}{2a_w^2}\right)\right] \\ &\quad \times\exp\left(-\sum_{j=1}^3 k_j^2l_j^2/2\right)\frac{1}{\boldsymbol{k}^2}(k_kk_p - \delta_{kp}k^2).\end{aligned}$$

As a result we obtain $\delta\nu_{kp}^{eff}=0$ for indices $k\neq p$. The results for $\delta\nu_{ii}^{eff}(\lambda)$ are given in Appendix A. The resulting effective magnetic permeability is shown in Fig. 2 for variance $q_0 = 0.1$ and for different correlation lengths, where we get $\delta\nu_{22}^{eff}=\delta\nu_{33}^{eff}$ for $l_2=l_3$. The graph shows the longitudinal and transversal permeability as a function of the scale λ. It is obvious that the asymptotic value strongly depends on the correlation lengths of the media. The effective value in the direction of the larger correlation length is higher than in the direction of the smaller correlation length, and the stronger the anisotropy the larger the difference between μ_{11}^{eff} and μ_{22}^{eff}.

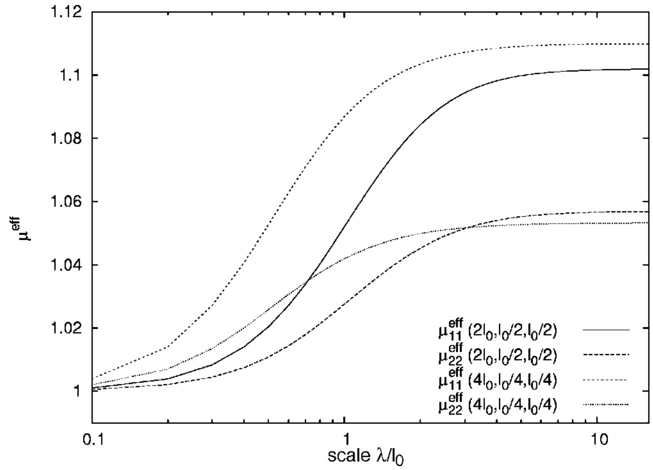

FIG. 2. Scale-dependent behavior of $\mu_{11}^{eff}(\lambda)$ and $\mu_{22}^{eff}(\lambda)$ as given by the leading-order approximation (A1) for the anisotropic case with $\bar{\nu}=1$, $q_0=0.1$, $a_w=2$, and correlation lengths $(2l_0,l_0/2,l_0/2)$ and $(4l_0,l_0/4,l_0/4)$.

V. DISCUSSION

Due to increasing experimental studies of materials and material phenomena reveiling new properties it is important to study theoretical approaches to model the underlying and scale-dependent material effects. To this end, a theoretical approach for the upscaling of material parameters is essential. However, based on the approximations the theoretical upscaling uses, it must be verified by numerical simulations or experimental data to prove its reliability. For the theoretical upscaling of flow and transport in heterogeneous media the applied coarse graining method proved primely its reliability according to small-scale and macroscopic numerical simulations, see, e.g., [13,19]. We would like to extend the numerical verification of the coarse graining method in the same fashion for the results obtained for the electromagnetic case. As a matter of fact we have to use an extended variational formulation of the equations of interest [see Eq. (B3) in Appendix B] in an appropriate Sobolev space. As Eq. (B3) includes the Laplace operator as well as the curl curl operator, a special treatment for the computation is due which turns out to be an inconvenient and extended finite element formulation (see also [16]). Nevertheless, we would like to perform a direct validation by numerical experiments in a future study. This validation will be analogous to the numerical verification of the upscaling of flow and transport in heterogeneous media where the results by the coarse graining formalism are well verified by the numerical simulations, see [13,19]. Currently, we are developing the appropriate discretization schemes to be able to compute the numerical upscaling, in particular, the permeability $\mu^{eff}(\lambda)$. It will enable us to compare the theoretical results of Sec. IV with numerical results.

The applications of the coarse graining method lie in the theoretical derivation of enhanced magnetic permeabilities. Such permeabilities arise in heterogeneous media given by, for instance, suspensions of solid particles with a high magnetic permeability in a liquid metal as discussed, e.g., experimentally in [20]. In this case the effective permeability is given as a function of the increasing volume fraction as measured in [20]. Other candidates for heterogeneous media are polymer composites, which also exhibit relatively high magnetic permeabilities, or turbulent fluids with macroscopic magnetic particles. The upscaling method can also be applied to calculate the effective magnetic properties of composite metamaterials consisting of periodically arranged circular conductive elements. The effective permeability as a function of the lattice width is an important quantity here. As shown in [21] the permeability strongly depends on the width of the computational grids used in numerical simulations. Many recent experimental studies have shown the possibility to create novel types of microstructured materials which demonstrate very interesting properties, see, e.g., [22]. Consequently, the upscaling results of the paper can be extended to cover the frequency dependence of the magnetic permeability, by evaluating $\delta\nu^{eff}(\lambda)$ for nonzero ω from Eq. (18) in the isotropic case or from Eq. (19) in the anisotropic case, as well as to be a function of the macroscopic volume fraction of magnetic particles in the media.

The understanding how the electric field depends on the resolution scale is also important to construct numerical models with coarser resolution scale. Here, upscaling methods may help to incorporate subgrid-scale information into the effective parameters of numerical models as given, e.g., in the recent work of Sterz [23]. For this reason, an application of the presented upscaling is the implementation of its results in computations using multigrid methods to improve the convergence efficiently. In the case of upscaling of flow, the numerical application of the upscaling and implementation in a multigrid method has already been done and has been proven very useful, see [24].

VI. CONCLUSION

We apply the coarse graining method to scale the electric field in the eddy current approximation of the Maxwell equations. The magnetic permeability of the material is assumed to have random heterogeneities given by a Gaussian random field. We derive an upscaled equation for the electric field which exhibits a scale-dependent effective permeability. The effective permeability is in general a tensor. We calculate the resulting magnetic permeability by a perturbation theory to obtain the dependence on the resolution scale. In the case of isotropic media the permeability reduces to a scalar value which increases with the scale and reaches its asymptotic value after ten correlation lengths. In the case of anisotropic correlation, the asymptotic values of the effective permeability coefficients depend strongly on the correlation lengths. Here, the larger the correlation length is the stronger the increase of the permeability as a function of the scale.

According to the derivation in the Appendixes, the upscaling method can be seen as an extension of the homogenization method towards the nonperiodic heterogeneous case where the upscaled permeability of the material is found from the solutions of local problems. The coarse graining method can therefore be applied to compute local upscaled magnetic permeabilities by numerical simulations.

A subsequent paper will be dedicated to the validation of the method and its results by numerical simulations using computer-generated heterogeneous media analogously to the numerical verification done in the work [13].

APPENDIX A: RESULTS FOR ANISOTROPIC CORRELATION

In the case of an anisotropic correlation function the result for the effective inverse permeability is given for $k=p=1$ by

$$\delta\nu_{11}^{eff}(\lambda) = \frac{q_0\bar{\nu}}{(2\pi)^{3/2}} l_1 l_2 l_3 \int_0^\infty d\Lambda \int \exp(-\Lambda \boldsymbol{k}^2)\left[1 - \exp\left(-\frac{\boldsymbol{k}^2\lambda^2}{2a_w^2}\right)\right]\exp\left(-\sum_{j=1}^{3} k_j^2 l_j^2/2\right)(k_2^2+k_3^2)d^3k$$
$$= \frac{q_0\bar{\nu}}{4\sqrt{2}} l_1 l_2 l_3 \int_0^\infty d\Lambda((\Lambda + l_1^2/2)^{-1/2}[(\Lambda + l_2^2/2)^{-3/2}(\Lambda + l_3^2/2)^{-1/2} + (\Lambda + l_2^2/2)^{-1/2}(\Lambda + l_3^2/2)^{-3/2}] - [\Lambda + \lambda^2/(2a_w^2) + l_1^2/2]^{-1/2}\{[\Lambda + \lambda^2/(2a_w^2) + l_2^2/2]^{-3/2}[\Lambda + \lambda^2/(2a_w^2) + l_3^2/2]^{-1/2} + [\Lambda + \lambda^2/(2a_w^2) + l_2^2/2]^{-1/2}[\Lambda + \lambda^2/(2a_w^2) + l_3^2/2]^{-3/2}\}). \quad \text{(A1)}$$

Analogously, we obtain $\delta\nu_{22}^{eff}$ by exchanging $l_1 \leftrightarrow l_2$ in Eq. (A1), and $\delta\nu_{33}^{eff}$ by exchanging $l_1 \leftrightarrow l_3$.

APPENDIX B: LOCAL UPSCALED PERMEABILITY IN REAL SPACE

In the case of an infinite length scale, i.e., $\lambda \to \infty$, the coarse graining method yields

$$\delta\nu_{ij}^{eff} = \epsilon_{ilm}\int_k\int_{k'} \overline{\hat{\tilde{\nu}}(-\boldsymbol{k}) i k_l \hat{G}_{mn}(\boldsymbol{k},-\boldsymbol{k}')\epsilon_{nqj} i k'_q \hat{\tilde{\nu}}(\boldsymbol{k}')}$$

which reduces due to the Fourier back transform to

$$\delta\nu_{ij}^{eff} = \epsilon_{ilm}\int \overline{\tilde{\nu}(\boldsymbol{x})\partial_{x_l} G_{mn}(\boldsymbol{x},\boldsymbol{x}')\epsilon_{nqj}\partial_{x'_q}\tilde{\nu}(\boldsymbol{x}')} dx' dx.$$

Consequently, the tensorial Green's function G_{im} fulfills for fixed index i (summing over m) the differential equation

$$-\bar{\nu}\sum_{j=1}^{3}\partial_j^2 \delta_{im} G_{im}(\boldsymbol{x},\boldsymbol{x}') + i\omega\sigma\delta_{im}G_{im}(\boldsymbol{x},\boldsymbol{x}') + \epsilon_{ijk}\epsilon_{klm}\partial_j\tilde{\nu}(\boldsymbol{x})\partial_l G_{im}(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}'). \quad \text{(B1)}$$

For finite length scales λ we obtain for the upscaled inverse permeability

$$\delta\nu_{ij}^{eff}(\lambda) = \epsilon_{ilm}\int \overline{\tilde{\nu}(\boldsymbol{x}) S(\boldsymbol{x}-\boldsymbol{x}',\lambda)\partial_{x'_l} G_{mn}(\boldsymbol{x}',\boldsymbol{x}'')\epsilon_{nqj} S(\boldsymbol{x}''-\boldsymbol{x}''',\lambda)\partial_{x'''_q}\tilde{\nu}(\boldsymbol{x}''')} d^3x''' \cdots d^3x$$

where the distribution S is defined by $S(\boldsymbol{x},\lambda) := \int_k \exp(i\boldsymbol{k}\cdot\boldsymbol{x}) P^+_{\lambda,k} = \delta(\boldsymbol{x}) - \Pi_{i=1}^d[\sin(x_i a_s/\lambda)/\pi x_i]$ according to the projection $P^+_{\lambda,k}$ in Fourier space. The expression for $S(\boldsymbol{x}-\boldsymbol{x}',\lambda)$ in $\delta\nu^{eff}(\lambda)$ is now simplified by an approximation similar to that in [13]. The integration over the convolution of S is approximated by a local smoothing $\int (S*f)(\boldsymbol{x}')d^3x' \approx \int_{\Omega_\lambda^{(x)}} f(\boldsymbol{x}')d^3x'$ ($\boldsymbol{x}$ fixed), which reproduces the exact integral for $\lambda=0$ and $\lambda\to\infty$. The smoothing is made over a volume $\Omega_\lambda^{(x)}$ proportional to λ^d where $\Omega_\lambda^{(x)}$ defines the d-dimensional cube $\Omega_\lambda^{(x)} := \Pi_{i=1}^d[x_i - \lambda/a_s, x_i + \lambda/a_s]$ with origin $\boldsymbol{x}$. Taking this approximation into account we obtain

$$\delta\nu_{ij}^{eff}(\lambda) = \epsilon_{ilm}\int \overline{\tilde{\nu}(\boldsymbol{x})\partial_{x_l}\int_{\Omega_\lambda^{(x)}} G_{mn}(\boldsymbol{x},\boldsymbol{x}')\epsilon_{nqj}\partial_{x'_q}\tilde{\nu}(\boldsymbol{x}')d^3x'} d^3x,$$

substituting the second term of S by the delta function. Further, we approximate Green's function $G_{mn}(\boldsymbol{x},\boldsymbol{x}')$ in $\delta\nu^{eff}$ by a local Green's function $G^{(x)}$ for $\Omega_\lambda^{(x)}$. To simplify the equation for the local Green's function we introduce an auxiliary function

$$\chi_{mj}^{(x)}(\boldsymbol{x}') := \int_{\Omega_\lambda^{(x)}} G_{mn}^{(x)}(\boldsymbol{x}',\boldsymbol{x}'')\epsilon_{nqj}\partial_{x''_q}\tilde{\nu}(\boldsymbol{x}'')d^3x''$$

where $\boldsymbol{x}' \in \Omega_\lambda^{(x)}$. Using Eq. (B1) this function $\chi_{im}^{(x)}(\boldsymbol{x}')$ fulfills an auxiliary equation in $\Omega_\lambda^{(x)}$

$$-\bar{\nu}\sum_{j=1}^{3}\partial_{x'_j}^2\delta_{in}\chi_{im}^{(x)}(\boldsymbol{x}') + i\omega\sigma\delta_{in}\chi_{im}^{(x)}(\boldsymbol{x}') + \epsilon_{ijk}\epsilon_{kln}\partial_j\tilde{\nu}(\boldsymbol{x}')\partial_l\chi_{im}^{(x)}(\boldsymbol{x}') = \epsilon_{nqm}\partial_{x'_q}\tilde{\nu}(\boldsymbol{x}')\delta_{in},$$

where the indices i and m are fixed. With the help of $\chi^{(x)}$ a local upscaled inverse permeability for $\Omega_\lambda^{(x)}$ is then given by

$$\delta\nu_{ij}^{eff}(\lambda,\boldsymbol{x}) = \epsilon_{ilm}\int_{\Omega_\lambda^{(x)}} \tilde{\nu}(\boldsymbol{x}')\partial_{x'_l}\chi_{mj}^{(x)}(\boldsymbol{x}')d^3x'. \quad \text{(B2)}$$

We remark that the function fulfills $\chi_{im}^{(x)}(\boldsymbol{x}') = \delta_{in}\chi_{nm}^{(x)}(\boldsymbol{x}')$ because of $\chi \equiv 0$ for $i \neq n$, and the auxiliary equation reads in real space for fixed i

$$-\bar{\nu}\Delta\delta_{in}\chi^{(x)}_{im}(\boldsymbol{x}') + i\omega\sigma\delta_{in}\chi^{(x)}_{im}(\boldsymbol{x}') + [\mathrm{curl}\ \tilde{\nu}(\boldsymbol{x}')\mathrm{curl}\ \chi^{(x)}_{\cdot m}(\boldsymbol{x}')\delta_{in}]_i = \delta_{in}\epsilon_{nqm}\partial_q\tilde{\nu}(\boldsymbol{x}'). \quad \text{(B3)}$$

As appropriate boundary conditions on $\partial\Omega^{(x)}_\lambda$ we propose $\epsilon_{ijk}n_j\chi^{(x)}_{km}=0$ for all $i,m\in\{1,2,3\}$ where n denotes the outer unit normal of $\partial\Omega^{(x)}_\lambda$. Thus Eq. (B3) can be solved numerically and due to Eq. (B2) $\delta\nu^{eff}_{ij}(\lambda,\boldsymbol{x})$ can be computed as a function of the scale and the location.

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